and for PA to be unchanged,

$$(\mathbf{P}-\mathbf{A})\cdot(\delta\mathbf{P}-\delta\mathbf{A})=0. \tag{8.3}$$

But since P, A, C, D are coplanar we may write

$$(\mathbf{P}-\mathbf{A})=\lambda(\mathbf{P}-\mathbf{C})+\mu(\mathbf{P}-\mathbf{D}). \tag{8.4}$$

Insert (8.4) in (8.3) and use (8.2) to get

$$(\mathbf{P}-\mathbf{A})\cdot\delta\mathbf{A}=0. \tag{8.5}$$

Since the possible displacements $\delta\mathbf{A}$ are perpendicular to the plane ABE, the condition (8.5) implies that P lies in this plane. It is now only a matter of geometry to determine what constraint this implies among the lengths BE, BD, CE and so derive the equation involving s, s_1, s_3 corresponding to the desired singularity. This is, of course, a tedious calculation.

Lastly, we give an example of the generation of a type (c) singularity by two different surfaces S_1, S_2. The simplest example is when S_1 is the surface corresponding to Fig. 11(a) and S_2 the surface corresponding to the contraction of Fig. 11(a) drawn in Fig. 11(b). The dual diagram for Fig. 11(a) is drawn in Fig. (12), which is a plane figure. That for Fig. 11(b) is similar, except that the line m_a is omitted and m_b, m_c are collinear. The required value of s_3, representing a singularity of the partial wave, is obtained by including m_a *and* making m_b, m_c collinear. This gives

$$s_3=m_a^2+m_b^2+(m_b/m_c)(m_a^2+m_c^2-M_2^2). \tag{8.6}$$

When (8.6) is satisfied the surface S_1, S_2 actually do more than touch in $(\cos\Theta,\Phi)$ space; they coincide.

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Legendre Transforms and Khuri Representations of Scattering Amplitudes*

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A representation of a scattering amplitude is described in which asymptotic behavior of the Regge type is exhibited in crossing symmetric form. It is based on Legendre transforms, which have similar meromorphy properties to partial wave amplitudes but use variables of the type $(s-2m^2)/2m^2$ instead of the cosine of the scattering angle. The representation obtained is another example of a class that has similar features to the crossing symmetric Sommerfeld-Watson transformation developed by Khuri and based on coefficients of a power series.

A REPRESENTATION that retains the crossing symmetry of the Mandelstam representation while incorporating the high-energy features of scattering amplitudes given by the Regge representation has been recently derived by Khuri.[1] His work helps to provide further justification of an approximation suggested earlier by Chew.[2] It is the purpose of this paper to note that a representation with similar characteristics to that of Khuri can be obtained from Legendre transforms of scattering amplitudes.[3] In particular, a lack of uniqueness is noted and it is suggested that this may give a valuable flexibility for the application of Khuri representations in practical calculations.

Legendre transforms[3] differ from partial wave amplitudes by the choice of variables of integration. For equal masses, define x_i by

$$s=2m^2(1+x_1),\quad t=2m^2(1+x_2),\quad u=2m^2(1+x_3), \tag{1}$$

with $x_1+x_2+x_3=-1$. In the first instance we assume that the Mandelstam representation holds without subtractions when $x_i<1$, $i=1$, 2, 3. Write the amplitude A in three parts corresponding to the three spectral regions, and consider one such part, A_{12} written $A'(x_1,x_2)$,

$$A'(x_1,x_2)=\frac{1}{\pi}\int_1^\infty\frac{dx_1'\,A_1'(x_1',x_2)}{x_1'-x_1}$$
$$=\frac{1}{\pi^2}\int_1^\infty\int_1^\infty\frac{dx_1'dx_2'\rho(x_1',x_2')}{(x_1'-x_1)(x_2'-x_2)}. \tag{2}$$

Define the single Legendre transform $B(l_1,x_2)$ by

$$B(l_1,x_2)=\frac{1}{\pi}\int_1^\infty dx_1'Q_{l_1}(x_1')A_1'(x_1',x_2), \tag{3}$$

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[1] N. Khuri, Phys. Rev. Letters **10**, 420 (1963).

[2] G. F. Chew, Phys. Rev. **129**, 2363 (1963).

[3] R. J. Eden (to be published).

with a similar equation for $B(x_1,l_2)$. It is convenient to take a slightly stronger condition than no subtractions in (2). We assume $A_1'(x_1',x_2)=O(x_1')^{-1/2}$, as $x_1'\rightarrow\infty$, for $|x_2|<1$. Hence, Eq. (3) defines a unique analytic continuation of $B(l_1,x_2)$ in the complex l_1 plane with $R(l_1)>-\frac{1}{2}$, and a Sommerfeld-Watson transform exists,

$$A'(x_1,x_2)=\frac{1}{2i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\frac{dl_1(2l_1+1)B(l_1,x_2)P_{l_1}(-x_1)}{\sin\pi l_1}. \quad (4)$$

If x_2 is varied so that a pole of $B(l_1,x_2)$ moves into $R(l_1)>-\frac{1}{2}$, this formula becomes modified by the addition of a term

$$b\{\alpha_1(x_2)\}P_{\alpha_1}(-x_1). \quad (5)$$

The pole at $l_1=\alpha_1$ with the largest real part has the same location as a pole in the usual partial wave amplitude. For a single pole in the latter, B will have a series of poles at $l_1=(\alpha_1-r)$, $r=0, 1, 2, \cdots$, and conversely.

Denote poles of $B(l_1,x_2)$, that enter the half-plane $R(l_1)>-\frac{1}{2}$, for any x_2 in its physical sheet, by the symbol α_{1r}. The assumption of a finite number of subtractions in the Mandelstam representation ensures that the α_{1r} are finite in number. Denote the corresponding set of poles of $B(x_1,l_2)$ by α_{2r}.

The function in (5), while it has the desired behavior for large x_1 when $R(\alpha_1)>-\frac{1}{2}$, has the undesired feature that it diverges for large x_1 when $R(\alpha_1)<-1$. In order to obtain an amplitude bounded in x_1 for all x_2 (in the physical sheet) we can subtract from $A'(x_1,x_2)$ *any* set of functions that vary like $(-x_1)^{\alpha_1}$ for large x_1 and $R(\alpha_1)>-\frac{1}{2}$ but are bounded in $R(\alpha_1)<-\frac{1}{2}$, provided we make a suitable choice of coefficients (which depend on x_2). In order to retain the Mandelstam representation for the remainder we require also that the functions have cut plane analyticity in x_1. These requirements do not specify the functions uniquely and this may provide a valuable flexibility in practical applications. Khuri[1] has given one suitable set of functions. Another is obtained by replacing (5) by

$$b(x_1,x_2)=b\{\alpha(x_2)\}\times\frac{\tan\alpha\pi}{\pi}\left[Q_{-\alpha-1}(-x_1)+\frac{1}{2}\int_{-1}^{1}\frac{P_\alpha(x)dx}{x+x_1}\right], \quad (6)$$

when $R(\alpha)>-1$, and by

$$b(x_1,x_2)=b\{\alpha(x_2)\}\times\frac{\tan\alpha\pi}{\pi}\left[\frac{\sin\alpha\pi}{\pi}\int_{1}^{\infty}\frac{Q_{-\alpha-1}(x)dx}{x-x_1}\right], \quad (7)$$

when $R(\alpha)<0$. In the region, $-1<R(\alpha)<0$, the two expressions are equivalent. If α passes through a half-integer the resulting pole should be subtracted from $b(x_1,x_2)$.

Write $b_r^{12}(x_1,x_2)$ for (6) and (7) when $\alpha=\alpha_{1r}$, and $b_r^{21}(x_2,x_1)$ for the analogous expression based on poles of $B(x_1,l_2)$. Define the amplitude $a_{12}(x_1,x_2)$ by

$$a_{12}(x_1,x_2)=A'(x_1,x_2)-\sum_r b_r^{12}(x_1,x_2)-\sum_r b_r^{21}(x_2,x_1). \quad (8)$$

In order to avoid ambiguity it is necessary to assume that the residues $b\{\alpha(x)\}=O(x^{-1/2})$ in the cut x plane; it is also assumed that they are analytic in this cut plane.

The Mandelstam representation holds without subtractions for $a_{12}(x_1,x_2)$, for any values of x_1, x_2 on the physical sheet,

$$a_{12}(x_1,x_2)=\frac{1}{\pi^2}\int_1^\infty\int_1^\infty\frac{dx_1'dx_2'\sigma(x_1',x_2')}{(x_1'-x_1)(x_2'-x_2)}. \quad (9)$$

The full amplitude A can now be expressed in terms of the Mandelstam representation and asymptotic terms,

$$A=a_{12}(x_1,x_2)+a_{23}(x_2,x_3)+a_{31}(x_3,x_1)+\sum_r[b_r^{12}(x_1,x_2)+b_r^{21}(x_2,x_1)+b_r^{23}(x_2,x_3)+b_r^{32}(x_3,x_2)+b_r^{31}(x_3,x_1)+b_r^{13}(x_1,x_3)]. \quad (10)$$

This is an example of another form of the Khuri representation.

From the asymptotic properties of $a_{12}(x_1,x_2)$ it is possible to deduce the existence of double Legendre transforms $c_{12}(l_1,l_2)$ which in this method are the analog of Khuri's expansion coefficients.[1] For example,

$$c_{12}(l_1,l_2)=\frac{1}{\pi^2}\int_1^\infty\int_1^\infty dx_1dx_2Q_{l_1}(x_1)Q_{l_2}(x_2)\sigma(x_1,x_2). \quad (11)$$

The amplitude $a_{12}(x_1,x_2)$ can be expressed in terms of $c_{12}(l_1,l_2)$ by a double Sommerfeld-Watson transform,

$$a_{12}(x_1,x_2)=-\frac{1}{4}\iint_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\frac{dl_1dl_2(2l_1+1)(2l_2+1)c_{12}(l_1,l_2)P_{l_1}(-x_1)P_{l_2}(-x_2)}{\sin\pi l_1\,\sin\pi l_2}. \quad (12)$$

The cut-plane analyticity of $a_{12}(x_1,x_2)$ is evident from the form of the integrand.

A more detailed account of the use of Legendre transforms will be published elsewhere.[3]